INTERACTION OF CYLINDRICAL MAGNET WITH SEMI-SPACE

V. Vorohobov
Institute of Physics, University of Latvia
32 Miera str., Salaspils, LV-2169, Latvia

An approach for calculation of the attraction force of permanent magnet to semi-space of a magnetizable material is developed. Simple analytical relations are proposed for engineering purposes, which guarantee 1%–10% accuracy at any distance ranging 0 to $+\infty$. This formula can be used for evaluation of a holding force created by the magnet, for measuring the magnetic moment and quality control of magnetization of cylindrical magnets, and for measuring the magnetic permeability $\mu$ of magnetic fluids and other substances, where $\mu$ in value is close to unity. Measurements of some magnetic fluid samples were taken. It is found experimentally that the accuracy is better when there is a standard gap between the magnet and the sample, which is equal to the radius of the magnet.

Experiments were carried out with magnets available on the market, for which a non-uniformity of magnetization about 5%–20% was found by measuring the sticking force in direct contact. A balance, which is able to measure very dissimilar forces, was created.

1. Theoretical calculation of the force. Let us consider the interaction of any magnet with the magnetizable semi-space with magnetic permeability $\mu$. The case when $\mu \to \infty$ is widely known, see, for example, [1], but it is also possible to use the reflection principle for any $\mu$.

To satisfy the boundary conditions

$$H_{\parallel 1} = H_{\parallel 2}, \quad B_{\perp 1} = B_{\perp 2},$$

here $B = \mu H$ that means

$$\frac{1}{\mu_1} B_{\parallel 1} = \frac{1}{\mu_2} B_{\parallel 2},$$

a fictitious magnet with magnetization

$$p = q \frac{1 - \mu}{1 + \mu}$$
must be placed in the reflected position.

The interaction force between the magnet and the magnetic material can be found as an interaction force between these two magnets: real and fictitious.

To find the force between the magnet and the plane, we have to solve the problem for two symmetrically equal magnets, and use a “reflection” coefficient

$$\frac{1 - \mu}{1 + \mu}.$$

The interaction force of permanents magnets can be calculated by analogy with current coils, Fig. 1.

Using the Biot–Savart law

$$d\mathbf{B} = \frac{i}{4\pi\varepsilon_0 c^2} \frac{d\mathbf{l} \times \mathbf{r}}{r^3}$$  \hspace{1cm} (1)
and an expression for the Ampere force on the current, we get the following relation for the force between two current loops. Only the International System of Units, where according to the definition $4\pi\varepsilon_0 c^2 = 10000000$, precisely is used in this paper. The expression $\varepsilon_0 c^2$ is used instead of $1/\mu_0$ following the tradition established in [1]. An approach allowing to derive the below formula can be found in [5] and [11]. The result is

$$f = \frac{(1 + 2w^2) E\left(\frac{1}{1+w^2}\right) - 2w^2 K\left(\frac{1}{1+w^2}\right)}{2\varepsilon_0 c^2 w \sqrt{1 + w^2}} N_t^2,$$

(2)

where $w = u/r$ is a relative distance between coils, showing how large is the distance in comparison with their diameters, and $N_t^2$ denotes ampere-turns.

Here,

$$K(q) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - q^2 \sin^2 \varphi}}$$

(3)

is the complete elliptic integral of the first kind, and

$$E(q) = \int_0^{\pi/2} \sqrt{1 - q^2 \sin^2 \varphi} d\varphi$$

(4)

is the complete elliptic integral of the second kind.

Following the approximation\(^1\) (2), it is possible that

$$f_{\text{approx}} \approx \frac{7.495287 \cdot 10^{-7}}{v(v + 1)(v^2 - v/2 + 1)} N_t^2,$$

(5)

where $v = 1.19293 \cdot w = 1.19293 \cdot (u/r)$

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\(^1\)The method for approximation of functions, which is used here, is different from the traditional mathematical approach by series used by pure mathematics and by computational mathematics as, for example, in [2], [9], [8]. The major idea of this method is to achieve a preciseness suitable for engineers, say $\pm 10\%$ for all values of arguments from $0$ to $+\infty$ by using any possible combinations of elementary functions. The simplest combination is considered as the best.
Interaction of cylindrical magnet with semi-space

![Diagram of cylindrical magnet with semi-space](image)

**Fig. 2.** The ratio between the precise formula (2) and approximation (3) of the force between the loops depending on the decimal logarithm $\lg(u/r)$.

This approximation coincides with the precise expression everywhere from 0 to $+\infty$ with the accuracy 5%. Fig. 2 displays the diagram allowing to compare the precise solution and the approximation.

Let us calculate the interaction force between two permanent magnets, Fig. 3.

To derive a formula, in this case we have to integrate formula (2) twice to sum the interaction of all pairs of indefinitely narrow currents. As soon as we speak about a current distributed over length $t$, we need to substitute $i \rightarrow (i/t)\,dL$ and integrate twice by $L = 2u$. Limits of the integration coincide with butts of the magnets. Here is the result of such integration:

$$
F = \frac{2N_i^2}{c^2\varepsilon_0 t^2} \left\{ u \sqrt{r^2 + u^2} \left[ K \left( \frac{r^2}{r^2 + u^2} \right) - E \left( \frac{r^2}{r^2 + u^2} \right) \right] \right. \\
- \frac{t + 2u}{2} \sqrt{4r^2 + (t + 2u)^2} \left[ K \left( \frac{4r^2}{4r^2 + (t + 2u)^2} \right) - E \left( \frac{4r^2}{4r^2 + (t + 2u)^2} \right) \right] \\
+ (t + 2u) \sqrt{r^2 + (t + u)^2} \left[ K \left( \frac{r^2}{r^2 + (t + u)^2} \right) - E \left( \frac{r^2}{r^2 + (t + u)^2} \right) \right] \left\} \right. \\
$$

where $u$ is the semi-distance between two coaxial solenoids measured between the nearest points, $t$ is the length of each of them, $N_i$ denotes ampere-turns in each solenoid, $r$ is the radius of the solenoids.

Using the analogy between the magnet and the current coils, it is possible to

![Diagram of magnets or solenoids](image)

**Fig. 3.** Two magnets or solenoids.
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derive an expression for attraction between a permanent cylindrical magnet and a
plane by substituting \( N_1 \to m/(r^2\pi) \to yt \), where \( m \) is the total magnetic moment
of the solenoid and \( y \) is the quantity of the magnetic moment per unit volume of
the magnet. Let us also introduce the “reflection coefficient” \( (1 - \mu)/(1 + \mu) \) to
yield the final basic expression:

\[
F(u) = \left( \frac{\mu - 1}{\mu + 1} \right) \frac{2y^2}{c^2\varepsilon_0} \left\{ u\sqrt{r^2 + u^2} \left[ K \left( \frac{r^2}{r^2 + u^2} \right) - E \left( \frac{r^2}{r^2 + u^2} \right) \right] - \frac{t + 2u}{2}\sqrt{4r^2 + (t + 2u)^2} \left[ K \left( \frac{4r^2}{4r^2 + (t + 2u)^2} \right) - E \left( \frac{4r^2}{4r^2 + (t + 2u)^2} \right) \right] \right\}.
\]

This formula for the force between the magnet and the plane is precise for any
parameters.

It is possible to derive a simple formula for some limiting cases.

1. At longer distances, when \( u \to \infty \),

\[
F = \frac{3\pi N^2 r^4}{32\varepsilon_0 c^2 u^4} = \frac{3m^2}{32\pi\varepsilon_0 c^2 u^4} = \frac{3\pi y^2 r^4 t^2}{32\varepsilon_0 c^2 u^4}.
\]

2. In case of a thin magnet and medium distances, when \( t \ll u \ll r \),

\[
F = \frac{N^2 r}{2\pi c^2 u} = \frac{m^2}{2\pi^2 c^2 r^3 u} = \frac{y^2 r t^2}{2\varepsilon_0 c^2 u}.
\]

3. Mathematically precise formula for the sticking force at full contact, when
\( u = 0 \), looks like

\[
F_{\text{stick}} \approx \left( \frac{\mu - 1}{\mu + 1} \right) \frac{r^2 y^2}{c^2\varepsilon_0} g \left\{ 2\sqrt{1 + g^2} \left[ K \left( \frac{1}{1 + g^2} \right) - E \left( \frac{1}{1 + g^2} \right) \right] - \sqrt{4 + g^2} \left[ K \left( \frac{4}{4 + g^2} \right) - E \left( \frac{4}{4 + g^2} \right) \right] \right\},
\]

where \( g = t/r \) denotes geometric proportions of the magnet, and \( y \) is the quantity
of the magnetic moment per unit volume of the magnet.

Here is the simple approximation for the sticking force (10):

\[
F_{\text{stick}} \approx \left( \frac{\mu - 1}{\mu + 1} \right) \frac{r^2 y^2 \pi}{c^2\varepsilon_0} \left[ 1.28389 \left( \frac{r}{t} \right)^2 + 0.2 \left( \frac{r}{t} \right) + 1 \right]^{1/4}.
\]

Note that \( s = r^2\pi \) is the contact surface area and
\[ 1.28389 = \frac{\pi}{2\ln(4)}. \]

This approximation coincides with the exact formula (10) with 1% accuracy. Such
accuracy is more than enough, because, as we will see later, the experiment always
gives a much larger error.

At full contact, when \( u = 0 \) and when magnets are very long \( g \to +\infty \),
formula (11) yields

\[
F_{\text{stick}} = \frac{1}{2c^2\varepsilon_0} y^2 S,
\]

where \( S = r^2\pi \) is the contact area between the magnets. Formula (12) is
appropriate not only for cylindrical geometry, but also for any other geometry. For
Fig. 4. The sticking force $F_{\text{stick}}$ given by formula (10) versus the geometric proportion $g = t/r$ (the radius of a neodymium magnet $r = 0.005$ m, its saturation magnetization $y = 915$ kA/m).

example, in [4], the same expression is derived for the interaction force between two semi-infinite parallelepipeds separated by a crack. The force tends to the limit given by (10)–(12) when the gap between the magnets becomes small $u \to 0$. Nevertheless, the derivative of the function $F(u)$ becomes unlimited at this point, and that is why it would be impossible to approximate this function by a Taylor serial around this point. Still some approximation can be done for the derivative:

$$\frac{\partial F}{\partial u}_{u \to 0} = \frac{1}{c^2 \varepsilon_0} y^2 2r \ln(u).$$

(13)

Herewith, it is obvious that the behaviour of the derivative depends only on the magnetization $y$ of one cubic meter of the magnets and on the perimeter of the crack between the magnets $l = 2\pi r$. It does not depend on the thickness of the magnet and either on the form of the magnet or contact surface area. Meanwhile the force itself depends mostly on the contact surface area (12), but not on the perimeter.

An indefinitely big in absolute value derivative near $u = 0$ makes measurements of the force difficult, and that is why this point is not the best for measurement of the magnetic moment of the magnets or measurement of the material properties.

Let us choose a convenient distance for measurements. On the one hand, the derivative must not be very big and, on the other hand, the force must be stronger than the force induced by the Earth magnetic field and the friction in the system. A reasonable compromise is possible at the distance $u = r$.

It is also possible to derive a precise formula for this case by substituting $u = r$ into (7), or it is possible to use the following approximation:

$$F_{u=r} \approx \left( \frac{\mu - 1}{\mu + 1} \right) \frac{y^2 r^2}{c^2 \varepsilon_0} \left[ 8.5 \left( \frac{r}{t} \right)^2 + 8.0 \left( \frac{r}{t} \right) + 6.81 \right]^{-1}.$$

(14)

Here $y$ is the quantity of the magnetic moment per unit volume of the magnet, $\mu$ is the permeability of the material, $r$ is the radius of the magnet, $t$ is the height of the magnet.

The accuracy of this approximation is 6%. A remarkable difference from the case of direct sticking (11) is the absence of the square root in the formula.
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In certain cases, the work function may be of interest. For example, the major reason of the damage of industrial lifting electromagnets is strong kicks by attracted iron pieces. It is emphasized also that neodymium alloy magnets, which are bigger than $0.05 \times 0.05 \times 0.05$ m$^3$, can be broken just by attracting iron. The energy of the kick is given by the work function.

To find the minimum energy $W$, which is needed to remove the magnet from the plane, it is necessary to integrate expression (7) from $u = 0$ to $u = +\infty$. The precise result of the integration is as follows

$$W = \left( \frac{\mu - 1}{\mu + 1} \right) \frac{g^2 r^3}{6c^2 \varepsilon_0} \left\{ 4 - 4(-1 + g^2)\sqrt{1 + g^2} E \left( \frac{1}{1 + g^2} \right) + (-4 + g^2)\sqrt{4 + g^2} E \left( \frac{4}{4 + g^2} \right) + 4g^2 \sqrt{1 + g^2} K \left( \frac{1}{1 + g^2} \right) - g^2 \sqrt{4 + g^2} K \left( \frac{4}{4 + g^2} \right) \right\},$$

(15)

where, as above, $g = t/r$, and $y$ is the quantity of the magnetic moment per unit volume of the magnet.

Some approximation was found also for this formula:

$$W = \left( \frac{\mu - 1}{\mu + 1} \right) \frac{r^3 y^2}{c^2 \varepsilon_0} \left[ \frac{2}{3} - \frac{8}{12 - 9g^2 \ln \left( 1 - \frac{128}{128 + 27g^2} \right)} \right].$$

(16)

This approximation coincides with the exact formula (15) for 10% for all parameters. However, for $t \gg r$ and for $t \ll r$ it is precise.

2. Experimental results. Helmholtz coils were used for calibration as a source of known uniform magnetic field. A permanent magnet hung in the center between two Helmholtz coils vibrates with a period $T$. A simple calculation yields the correct value of the field at the center point. With the radius and the distance being $R$, the number of turns in each coil $N$ and the current, passing through the coils, $i$, the magnetic flux density, $B$ at the midpoint between the coils is given by

$$B = \left( \frac{4}{5} \right) \frac{N i}{\varepsilon_0 c^2 R}.$$

(17)

In our system, which has been produced specially for this measurement, the coils are made from !UNITS!: 0.0002 diameter wire, $R = 0.0825$, $N = 200$ in each coil, so we get $B = 0.002170i$. The total magnetic field includes also the Earth magnetic field. To exclude its influence, we take two measurements with Helmholtz coils of magnetic fields being oriented pro and contra the Earth magnetic field. Then the measurements were repeated using only the Earth magnetic field. It provides also an option to calculate the horizontal component of the Earth magnetic field.

The mechanical moment of the cylinder is calculated by the formula

$$J_{\text{cylinder}} = M \left( \frac{t^2}{12} + \frac{r^2}{4} \right),$$

(18)

where $M$ is the mass of the magnet.
### Table 1.

<table>
<thead>
<tr>
<th>Unit</th>
<th>Nd magnet</th>
<th>CoFe 1</th>
<th>CoFe 2</th>
<th>FeBa</th>
<th>Magnetic rubber</th>
<th>CoFe 1+2</th>
</tr>
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<tbody>
<tr>
<td>$T_-$ sec</td>
<td>0.40275</td>
<td>0.6613</td>
<td>0.731</td>
<td>0.42</td>
<td>0.538</td>
<td></td>
</tr>
<tr>
<td>$T_+$ sec</td>
<td>0.3835</td>
<td>0.615</td>
<td>0.685</td>
<td>0.375</td>
<td>0.534</td>
<td></td>
</tr>
<tr>
<td>$T_0$ sec</td>
<td>1.386</td>
<td>2.45</td>
<td>2.76</td>
<td>1.123596</td>
<td>0.533</td>
<td>3.964</td>
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<tr>
<td>Current in Helmholtz coils</td>
<td>A</td>
<td>0.094</td>
<td>0.096</td>
<td>0.096</td>
<td>0.05</td>
<td>0</td>
</tr>
<tr>
<td>M weight kg</td>
<td>0.0195</td>
<td>0.0364</td>
<td>0.03677</td>
<td>0.00209</td>
<td>0.00051</td>
<td>0.0728</td>
</tr>
<tr>
<td>Length meter</td>
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<td>0.0175</td>
<td>0.0175</td>
<td>0.00253</td>
<td>0.00275</td>
<td>0.035</td>
</tr>
<tr>
<td>Radius meter</td>
<td>0.005</td>
<td>0.00075</td>
<td>0.00075</td>
<td>0.0075</td>
<td>0.0045</td>
<td>0.00975</td>
</tr>
<tr>
<td>$J$, mechanical moment N·m</td>
<td>1.89 · 10^{-6}</td>
<td>1.79 · 10^{-6}</td>
<td>1.81 · 10^{-6}</td>
<td>3.05 · 10^{-8}</td>
<td>2.903 · 10^{-9}</td>
<td>9.162 · 10^{-6}</td>
</tr>
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<td>$B_{\text{Helmholtz}}$ Gaus</td>
<td>0.000204</td>
<td>0.000208</td>
<td>0.000208</td>
<td>0.000109</td>
<td>0</td>
<td>0.0007465</td>
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<tr>
<td>$B_{\text{Earth}}$ Gaus</td>
<td>1.64E-05</td>
<td>1.41E-05</td>
<td>1.37E-05</td>
<td>1.34E-05</td>
<td>1.365E-05</td>
<td></td>
</tr>
<tr>
<td>Average Earth field m, magnetic moment A·m^2</td>
<td>0.0000145</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$F$ theoretical N</td>
<td>40.18858</td>
<td>3.928725</td>
<td>2.64189</td>
<td>0.77891</td>
<td>0.1158299</td>
<td>4.5460951</td>
</tr>
<tr>
<td>$F$ experimental first side N</td>
<td>35</td>
<td>4.6</td>
<td>4.6</td>
<td>1.2</td>
<td>0.12</td>
<td>5.5</td>
</tr>
<tr>
<td>$F$ experimental second side N</td>
<td>40</td>
<td>4.8</td>
<td>3.6</td>
<td>1</td>
<td>0.13</td>
<td>4.5</td>
</tr>
<tr>
<td>Deviation of magnetization degree from the axis</td>
<td>&lt; 5</td>
<td>20</td>
<td>20</td>
<td>&lt; 5</td>
<td>&lt; 5</td>
<td>10</td>
</tr>
</tbody>
</table>
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If \( J \) is the total moment of inertia of the magnet, then the total dipole magnetic moment \( m \) of the magnet can be calculated as

\[
m = \frac{J}{B \pm B_{\text{Earth}}} \left( \frac{2\pi}{T} \right)^2 \]

It means that

\[
m = \frac{J}{2B} \left[ \left( \frac{2\pi}{T_+} \right)^2 + \left( \frac{2\pi}{T_-} \right)^2 \right] \tag{19}
\]

and

\[
B_{\text{Earth}} = \frac{J}{m} \left( \frac{2\pi}{T_0} \right)^2 = 2B \frac{T_0^{-2}}{T_+^{-2} + T_-^{-2}}. \tag{20}
\]

Six magnets were used in the experiment. The Table 1 presents results of the measurements of magnetic moments of the six different magnets and the force of their sticking to the iron (both theoretically calculated and experimentally found).

So as seen from the table, the non-uniformity of magnetization of real magnets can be sufficiently large.

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Fig. 5. Measurement of the magnetic permeability of fluids using the simple balance.
Interaction of cylindrical magnet with semi-space

\[ \log(F) \] [N]

-1.5
-2.0
-1.0
-0.5
0.0
0.5
1.0

\( u [\text{m}] \)
0.005
0.010
0.015
0.020

Fig. 6. Decimal logarithm of the force \( F \) of attraction to iron versus the distance \( u \). The solid line is given by relation (7) at \( \mu \to \infty \), \( m = 0.151 \text{ m}^2\text{A} \), and filled rhombuses show the experimental data for clean iron.

Simple balance was created to measure the attraction force between the cylindrical magnet and the plane, Fig. 6. Moving the central bar (rule) you must find its position when the magnet comes off. You can use different bars with different weights: heavy cooper or light bamboo – that is why this balance works in a wide range of forces. This is its advantage if compared with the Rankine balance [4] suitable only for weak forces or regular scales suitable only for big forces. The distance between the magnet and the material can be varied by gaskets. This balance may be used to measure the permeability of magnetic fluids because it is even possible to take measurements without taking liquid from the bottle.

To measure the permeability \( \mu \), it is possible to use any magnet, not cylindrical. However, the use of a cylindrical magnet is more convenient and precise. Two forces have to be measured under equal conditions: attraction to a plain sample and attraction to iron. Their ratio gives \( (1 - \mu)/(1 + \mu) \). To take the measurements, the calibration is necessary. For this purpose, the magnet attraction to the plane of clean iron made by the American Rolling Mill Corporation (ARMCO) was used. The permeability of clean iron is big and hence \( (\mu - 1)/(\mu + 1) = 1 \).

For the geometry of our device, \( r = 0.0053 \text{ m}, t = 0.00203 \text{ m}\). By choosing different options for the magnetic moment, it is found experimentally that the value \( m = 0.151 \text{ m}^2\text{A} \) fits best to get agreement between theoretical and experimental data.

The goal was to find the coefficient \( (\mu - 1)/(\mu + 1) \) in our theoretical formula. In Fig. 7, one can see a graph, where it is assumed that \( (\mu - 1)/(\mu + 1) = 0.23 \) (in other words, \( \mu = 1.6 \)).

It is observed that our assumption for the permeability works very well for distances comparable with the radius of the magnet and longer. But at shorter distances, the experimental results for the force are sufficiently less. The only explanation of this fact can be the dependence of the magnetic permeability of the magnetic fluid on the magnetic field strength.

Several reasons contribute to the errors of the measurements.

1. One of the major sources of errors comes from the uncertain distance between the magnet and the surface of the tested material. If the surface is smooth and polished, we achieve a preciseness of about 5% for \( (\mu - 1)/(\mu + 1) \). But for the most of real materials, the surface is not very good - it is either corrugated or curved. Therefore, an error of 20–30% is possible. The magnetic field there is not uniform, that is why our method is sensitive to the surface quality. In most cases,
it is possible to improve the preciseness by using a bigger or a smaller magnet: bigger than local defects, but smaller than the general curvature or the size of the material.

2. A regular error can come from incorrect magnetization of the magnets. As we observe above, the error, which is caused by this factor, can be about 20%.

3. For materials with small \((\mu - 1)/(\mu + 1)\), the error is determined by the Earth magnetic field. Therefore, it is better to use a strongest magnet and place it as close as possible. Another method to avoid this influence is to repeat the measurements turning over the magnet.

4. Some materials like magnetic fluids have nonlinear magnetization. This fact restricts our possibility to use strong magnets in close position. Some optimum can be found, for example, for our system the optimal distance is about 5 mm.

So it has been found in our experiment that the preciseness is better when there is a standard gap, which is equal approximately to the radius of the magnet \(u = r\), i.e., in the case when formula (14) can be used.

The proposed relations can be applied for measuring the thickness of enamel using formula (7) and a calibrated small magnet with additional weight, Fig. 8. Such magnet can be calibrated by measuring the attraction force.

By moving this plastic bar right and left, it is possible to find its position when the magnet slightly goes away from the iron. Knowing all distances and
the bar’s and magnet’s weights, it is possible to calculate the force of attraction. Then formula (14) must be used. For example, for a neodymium alloy magnet we get \( y = 980 \text{kA/m}, m = 2.54 \text{m}^2 \text{A} \); for a CoFe alloy magnet \( y = 215 \text{kA/m}, m = 1.12 \text{m}^2 \text{A} \), with a possible error +10%.

3. Conclusion. The initial goal of this work was to calculate the holding force created by a real magnet. But during the study, more possible applications were found such as measurements of magnetization and quality control of magnets, measurements of the magnetic permeability, examination of the thickness of enamel, evaluation of energy of destructive kicks (work function).

The balance proposed here can find application for measurements of any strongly dissimilar forces. The express method of calibration of magnets can be used also as an inexpensive lab work for students.

4. Acknowledgements. The author thanks Prof. A. Cēbers and K. Ėrglis for assistance and O. Petrichenko for the sample of the magnetic fluid.

REFERENCES


Received 08.01.2010

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